

# Yang-Mills theories on the space-time $S_1 \times R$ cylinder: equal-time quantization in light-cone gauge and Wilson loops

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## Abstract

Pure Yang-Mills theories on the  $S_1 \times R$  cylinder are quantized in light-cone gauge  $A_- = 0$  by means of **equal – time** commutation relations. Positive and negative frequency components are excluded from the “physical” Hilbert space by imposing Gauss’ law in a weak sense. Zero modes, related to the winding on the cylinder, provide non trivial topological variables of the theory. A Wilson loop with light-like sides is studied: in the abelian case it can be exactly computed obtaining the expected area result, whereas difficulties are pointed out in non abelian cases.

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## I. INTRODUCTION

Yang-Mills theories on the two-dimensional space-time cylinder  $S_1 \times R$  have been extensively studied in the recent literature [1–4]. The common wisdom says that, without fermions, no dynamical degrees of freedom are available in two dimensions, but only topological excitations which, to be studied, need (at least a partial) compactification of the base manifold. As we have in mind the possibility of studying Wilson loops with light-like sides, we shall strictly comply with a Minkowski formulation and put our system in a spatial box with periodic boundary conditions. The system will be allowed to evolve in time according to its peculiar dynamics. As for the fiber, we shall confine ourselves to  $SU(N)$ , although the treatment might be further generalized. In passing, also the simpler case of QED will be discussed. We shall consider pure Yang-Mills theories, deferring to a future investigation the introduction of dynamical fermions, which would entail important consequences on the Hilbert space structure of the theory.

Quantization has been performed in the literature using Coulomb or Hamilton gauges on the group algebra [2]; the toroidal structure of the system has been fully clarified, together with the properties of Gribov’s copies [3]. In particular the energy levels and the Hilbert space of states have been obtained.

We should also mention the possibility of quantizing the system on the group manifold itself (and not on its algebra) [1]. In this case one can avoid gauge fixing, but the spectrum one gets is different from the one obtained in the previous approach by the Casimir “energy” due to the group curvature [4].

In the following we shall restrict ourselves to the group algebra approach. We are interested in learning what are the differences occurring when the system is quantized in non-covariant light-cone gauge. As is well known, this gauge has advantages and difficulties on its own. First of all it is not allowed to compactify the base manifold in the gauge vector direction; in the case we are studying the condition  $A_1 = 0$  would lead to inconsistencies. No contradiction will instead ensue from the choice  $A_- = 0$ ; we shall indeed refrain from

using light-front quantization: to compactify the range of  $x^-$  would not be allowed without drastically changing the gauge condition (*e.g.* by choosing  $\partial_- A_- = 0$ ). We shall quantize our system according to equal time commutation relations.

There are several reasons supporting this choice: from a physical view point a system naturally evolves in **time**. To force evolution in  $x^+$  entails the appearance of unnecessary zero modes which in turn generate “artificial” infrared singularities. On the other hand no simplification would ensue, as we want to put the system in a spatial box  $-L \leq x \leq L$ , with periodic boundary conditions.

At variance with the Coulomb gauge choice, the Faddeev-Popov determinant is irrelevant in this case.

Our interest in light-cone gauge on a cylinder is prompted by some results we have found in the continuum in strictly  $1 + 1$  dimensions [5]. To summarize them, we recall that different finite expressions are obtained for a rectangular Wilson loop with light-like sides, of “lengths”  $\lambda$  and  $\tau$  respectively, according to different choices for the vector propagator. The first choice has the “contact” form of a real delta-like interaction concentrated at the value  $x + t = 0$  (see *Sect.III*). It has been suggested by ’t Hooft [6] and used in performing the calculation of the quark propagator and of the mesonic wave functions in the large- $N$  limit. However, as was already noticed at that time [7], this propagator exhibits extra singularities under Wick’s rotation.

Canonical equal time quantization in the continuum suggests instead a different expression for the propagator in which massless excitations contribute in a causal way, although with negative probability (ghosts) [8,5]. They give rise indeed to a complex “potential” with an absorptive part, which possesses smooth properties under Wick’s rotation. This form can also be obtained as the two-dimensional limit of the  $d$ -dimensional Mandelstam-Leibbrandt (M-L) propagator, that turns out to be essential in order to prove the renormalizability of the theory in  $d = 3 + 1$  [9]. Precisely this propagator is also required to recover gauge invariance of the Wilson loop in  $(3 - \epsilon) + 1$  dimensions [10].

In turn both Wilson loop results are different from the gauge invariant expression one

obtains when the theory is considered in  $(1 + \epsilon) + 1$  dimensions and the limit  $\epsilon \rightarrow 0$  is eventually performed [5]. In our opinion this phenomenon reflects a basic “discontinuity” of the theory in the limit  $\epsilon \rightarrow 0$ , rather than rising doubts on light-cone gauge.

As the calculation in strictly 1+1 dimensions would exhibit in the continuum infrared divergences when choosing Feynman or Coulomb gauges, at least in intermediate steps, and thereby would prevent a direct comparison with light-cone gauge results, as a first step we feel interesting a careful understanding of the theory on a cylinder when quantized in light-cone gauge imposing equal *time* canonical commutation relations.

This is done in *Sect.II*, paying particular attention both to zero modes and to positive and negative frequency excitations. Canonical quantization, Gauss’s law, Hilbert space of states and eigenvalues of the hamiltonian are carefully discussed for the algebras  $su(2)$  and  $su(3)$ . In particular the spectrum of topological excitations coincides with known results obtained in Coulomb gauge. In *Sect.III* a particular Wilson loop with light-like sides is studied: in the abelian case it can be exactly computed obtaining the expected area result, whereas several difficulties are pointed out preventing a full solution in the non abelian cases and rising serious criticism concerning perturbative approaches.

Final conclusions are drawn and possible future developments are suggested in *Sect.IV*.

## II. EQUAL-TIME QUANTIZATION IN LIGHT-CONE GAUGE

We start from the lagrangian density

$$\mathcal{L} = -\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu}) - 2Tr(\lambda nA). \quad (1)$$

$F_{\mu\nu}$  is the usual field tensor,  $A_\mu$  the vector potential,  $\lambda$  is a Lagrange multiplier and the gauge vector  $n_\mu$  is lightlike. The usual normalization

$$Tr(T_a T_b) = \frac{1}{2}\delta_{ab} \quad (2)$$

is understood. We introduce  $\pm$  components and, to be specific, we choose  $n_+ = 1$ . In the following a “conjugate” lightlike vector  $n_-^* = 1$  will also be considered.

From the lagrangian (1) one can immediately derive the Euler equations

$$\begin{aligned} D^\mu F_{\mu\nu} &= \lambda n_\nu, \\ nA &= 0, \end{aligned} \tag{3}$$

$D^\mu$  being the covariant derivative acting on the adjoint representation. Eqs.(3) can be rewritten as

$$\partial_-^2 A_+ = 0, \tag{4}$$

$$\partial_- \partial_+ A_+ - ig[A_+, \partial_- A_+] = \lambda, \tag{5}$$

$$A_- = 0. \tag{6}$$

The Lagrange multiplier is equal to Gauss' operator and satisfies the equation

$$\partial_- \lambda = 0 \tag{7}$$

as a consequence of eqs.(4)-(6).

We consider our theory on a space-time cylinder, the space variable  $x$  taking values between  $-L \leq x \leq L$  and all quantities satisfying periodic boundary conditions; we stress that the time variable  $t$  is not compactified. The possibility of considering the theory on a cylinder is not trivial when imposing axial gauges: for instance the gauge fixing  $A_1 = 0$  would not be allowed, as is well known.

It is natural to expand the potential in a Fourier series

$$A_+(t, x) \equiv A(t, x) = \frac{1}{\sqrt{2L}} \sum_{k=-\infty}^{\infty} A_k(t) \exp(i \frac{\pi k x}{L}), \tag{8}$$

$$A_k(t) = \frac{1}{\sqrt{2L}} \int_{-L}^L A(t, x) \exp(-i \frac{\pi k x}{L}) dx. \tag{9}$$

Eq.(4) has the solution

$$A_k(t) = (a_k t + b_k) \exp(i \frac{\pi k t}{L}) \tag{10}$$

and therefrom

$$A(t, x) = \frac{1}{\sqrt{2L}} \sum_{k=-\infty}^{\infty} (a_k t + b_k) \exp(i \frac{\pi k(t+x)}{L}) = \frac{1}{\sqrt{2L}} (b_0 + a_0 t) + \hat{A}(t, x). \quad (11)$$

Similarly eq.(7) has the solution

$$\lambda(t, x) = \frac{1}{\sqrt{2L}} \sum_{k=-\infty}^{\infty} \lambda_k \exp(i \frac{\pi k(t+x)}{L}). \quad (12)$$

In this treatment  $A$ ,  $F \equiv F_{-+}$  and  $\lambda$  look essentially as free fields, all dynamics being transferred to the Gauss' law (eq.(5))

$$\lambda_k = \frac{i\pi k}{L} a_k - \frac{ig}{2\sqrt{L}} \sum_{p,q=-\infty}^{\infty} [b_p, a_q] \delta_{k,p+q}. \quad (13)$$

One can also easily derive the hamiltonian corresponding to (1)

$$H = \int_{-L}^L dx [Tr(F^2 - A\lambda)] = \sum_{k=-\infty}^{\infty} [Tr(a_k a_{-k}) - \frac{i\pi}{L} k Tr(a_k b_{-k})]. \quad (14)$$

When  $k \neq 0$  it is natural to introduce positive and negative frequencies; we define for  $k > 0$

$$b_k^{\pm} = b_{\pm k}, \quad \pm i a_k^{\pm} = a_{\pm k}, \quad (15)$$

with the conjugation properties

$$(b_k^{\pm})^{\dagger} = b^{\mp}, \quad (a_k^{\pm})^{\dagger} = a^{\mp}. \quad (16)$$

The classical hamiltonian takes the form

$$H = Tr(a_0^2) + 2 \sum_{k=1}^{\infty} Tr(a_k^+ a_k^-) + \frac{\pi}{L} \sum_{k=1}^{\infty} k Tr(a_k^+ b_k^- + b_k^+ a_k^-). \quad (17)$$

Similarly eq.(12) can be written as

$$\lambda = \frac{1}{\sqrt{2L}} [\lambda_0 + \sum_{k=1}^{\infty} (\lambda_k^+ \exp(i \frac{\pi k(t+x)}{L}) + \lambda_k^- \exp(-i \frac{\pi k(t+x)}{L}))]. \quad (18)$$

Thanks to eq.(7), frequencies do not mix under time evolution. Moreover we get

$$\lambda_0 = -\frac{ig}{2\sqrt{L}} [b_0, a_0] + \frac{g}{2\sqrt{L}} \sum_{p=1}^{\infty} ([b_p^-, a_p^+] - [b_p^+, a_p^-]), \quad (19)$$

$$\begin{aligned}
\lambda_k^- &= -\frac{\pi k}{L} a_k^- - \frac{ig}{2\sqrt{L}} [b_k^-, a_0] - \frac{g}{2\sqrt{L}} [b_0, a_k^-] + \frac{g}{2\sqrt{L}} \sum_{p=1}^{\infty} [b_{p+k}^-, a_p^+] - \\
&\quad - \frac{g}{2\sqrt{L}} \sum_{p=1}^{k-1} [b_{k-p}^-, a_p^-] - \frac{g}{2\sqrt{L}} \sum_{p=k+1}^{\infty} [b_{p-k}^+, a_p^-], \\
k &> 0, \quad \sum_{p=1}^0 = 0,
\end{aligned} \tag{20}$$

and

$$\lambda_k^+ = (\lambda_k^-)^\dagger. \tag{21}$$

We notice that the hamiltonian describes a free theory; it does not depend on the coupling constant  $g$  and does not mix Fourier components. Actually the dependence on  $g$  is buried in the expression of the Gauss' operator.

The canonical Poisson brackets for our system are [8]

$$\{A^r(t, x), F^s(t, y)\} = \sqrt{2} \delta^{rs} \delta(x - y), \tag{22}$$

the Dirac distribution  $\delta$  being understood as its periodic generalization

$$\delta_P(x - y) = \frac{1}{2L} \sum_{k=-\infty}^{\infty} \exp\left[\frac{i\pi k}{L}(x - y)\right]. \tag{23}$$

Eq.(22) entails

$$\{b_k^r, a_j^s\} = \delta^{rs} \delta_{k, -j}, \tag{24}$$

all other brackets vanishing.

For  $k \neq 0$  the algebra (24) becomes

$$\{b_k^{r\mp}, a_j^{s\pm}\} = \mp i \delta^{rs} \delta_{kj}. \tag{25}$$

At the classical level we can impose the following residual gauge condition on zero modes

$$b_0^\alpha \simeq 0, \tag{26}$$

for any  $\alpha$  not belonging to the Cartan subalgebra. Here  $\simeq$  denotes weakly equal in Dirac's terminology, as the constraints (26) are not compatible with the canonical brackets (24).

Consistency under time evolution of (26) entails

$$a_0^\alpha \simeq 0, \quad (27)$$

for the same values of  $\alpha$ . Together they give rise to a couple of “second class” constraints, that will be enough to get the vanishing of the first term in the right hand side of eq. (19), in partial fulfilment of the Gauss’ law. To impose the vanishing of all the zero modes would neither be requested nor be allowed, as it will become clear in the following. We should remark that zero modes do not contribute in the naive decompactification limit  $L \rightarrow \infty$ . Consequently, in such a limit, our treatment smoothly tends to the one developed in [5], in spite of the partial use of Gauss’ law we made to cast  $b_0$  as in (26).

Obviously constraints (26),(27) are incompatible with the Poisson brackets (24) for  $k = 0$ . Those brackets have to be modified into Dirac brackets, according to the equation

$$\{b_0^r, a_0^s\}_{\mathcal{D}} = \{b_0^r, a_0^s\} - \{b_0^r, a_0^\alpha\} \{a_0^\alpha, b_0^\beta\} \{b_0^\beta, a_0^s\}, \quad (28)$$

where  $\alpha$  and  $\beta$  do not belong to the Cartan subalgebra.

At a classical level Gauss’ law is to be imposed as a constraint on the allowed trajectories of the system.

Quantization is performed turning Dirac brackets into quantum commutators. As a consequence zero mode operators not belonging to the Cartan subalgebra vanish, whereas zero mode operators belonging to the Cartan subalgebra satisfy canonical commutation relations.

A Fock vacuum  $|\Omega\rangle$  will then be defined as the state annihilated by  $a_k^-$  and  $b_k^-$  for any positive  $k$ .

We explicitly remark the possibility of having negative norm states; the excitations we are quantizing have a ghostlike character [8] and therefore cannot be present in the Hilbert space of “physical” states. They will be extruded by imposing the Gauss’ law fulfilment.

The Gauss’ law will be imposed at the quantum level in an average sense by means of the weak condition

$$\langle \Psi_{phys} | \lambda^r(t, x) | \Phi_{phys} \rangle = 0, \quad (29)$$

which is stable under time evolution, the “physical” states belonging to a Hilbert space  $\mathcal{H}_{phys}$  with positive semidefinite metric [8]. Eq.(29) can be Fourier decomposed, leading to

$$\langle \Psi_{phys} | \lambda_0 | \Phi_{phys} \rangle = 0, \quad (30)$$

and

$$\lambda_k^- | \Phi_{phys} \rangle = 0, \quad \forall k > 0. \quad (31)$$

Eqs.(30) and (31) can in turn be realized by requiring

$$a_k^- | \Phi_{phys} \rangle = 0, \quad b_k^- | \Phi_{phys} \rangle = 0, \quad \forall k > 0. \quad (32)$$

The hamiltonian, when restricted to the “physical” subspace, just becomes

$$H_{phys} = Tr(a_0^2) \quad (33)$$

and we are left with only zero modes belonging to the Cartan subalgebra.

In the following we shall focus our interest on  $su(2)$  and on  $su(3)$ .

In  $su(2)$  we choose  $b_0$  along  $\sigma^3$ :  $b_0 = \beta_3 \frac{\sigma^3}{2}$ . At a classical level, we have the possibility of performing the residual gauge transformation

$$\begin{aligned} [U(t+x)] &= \exp\left[\frac{i\pi}{L}(t+x)\sigma^3\right] \\ &= \cos\left[\frac{\pi}{L}(t+x)\right] + i\sigma^3 \sin\left[\frac{\pi}{L}(t+x)\right], \end{aligned} \quad (34)$$

which is globally defined on the cylinder. It induces on  $\beta_3$  the transformation

$$\beta_3 \rightarrow \beta_3 + \frac{4\pi}{g\sqrt{L}} \quad (35)$$

suggesting the introduction of an angular variable  $\theta$  according to the equation

$$\beta_3 = 2\theta(g\sqrt{L})^{-1}, \quad -\pi \leq \theta \leq \pi. \quad (36)$$

The operator  $U$ , which generates the translation of  $\beta_3$  of a period, has the following quantum representation

$$U = \exp\left[\frac{4\pi i}{g\sqrt{L}}\alpha_3\right], \quad (37)$$

where  $a_0 = \alpha_3 \frac{\sigma^3}{2}$ . The canonical algebra suggests

$$\alpha_3 = -i \frac{g\sqrt{L}}{2} \frac{d}{d\theta}. \quad (38)$$

The corresponding spectrum of the hamiltonian in eq.(33), when restricted to the “physical” subspace, turns out to be [2]

$$E_n = \frac{g^2 L}{8} n^2, \quad (39)$$

(see eq.(38)).

We remark that, apart from the fundamental level  $n = 0$ , all other energy eigenvalues are linearly increasing with  $L$  and would diverge in the decompactification limit. They can be generated by acting on the fundamental level by the operator  $\exp[i\theta]$ , which in turn is nothing but the average on the state  $|\Omega\rangle$  of the Wilson loop wrapping once around the cylinder at  $t = 0$ .

A unique vacuum with respect to the Hamiltonian corresponds to  $n = 0$  [2]. The situation might be quite different in the presence of dynamical fermions.

The space of states can be taken as the direct product of the Fock space related to non vanishing frequencies times the Hilbert space of zero modes  $\mathcal{R}$ .

The “physical” Hilbert space  $\mathcal{H}_{phys}$  is realized as the product  $|\Omega\rangle \times \mathcal{R}$ . Nevertheless we shall discover that the ghost-like degrees of freedom, although excluded from  $\mathcal{H}_{phys}$ , entail important consequences even on gauge invariant quantities (typically Wilson loops).

It is now instructive to see how this treatment is generalized to  $\mathfrak{su}(3)$ .

Let us expand  $b_0$  on the Cartan subalgebra of  $\mathfrak{su}(3)$  that we parametrize according to the Gell-Mann basis

$$b_0 = \beta_3 T^3 + \beta_8 T^8. \quad (40)$$

Then, following eq.(27) it is natural to decompose

$$a_0 = \alpha_3 T^3 + \alpha_8 T^8. \quad (41)$$

We have still the possibility of performing residual gauge transformations of the kind

$$U(t+x) = \exp\left[\frac{2i\pi}{L}(t+x)n_3 T^3\right] \exp\left[\frac{i\pi}{L}(t+x)n_8(T^3 + \sqrt{3}T^8)\right], \quad (42)$$

which are globally defined on the cylinder as long as  $n_3$  and  $n_8$  take integral values.

As a consequence

$$\begin{aligned} \theta_3 &= \frac{g\sqrt{L}}{2}(\beta_3 - \frac{1}{\sqrt{3}}\beta_8), \\ \theta_8 &= \frac{g\sqrt{L}}{\sqrt{3}}\beta_8 \end{aligned} \quad (43)$$

turn out to be angular variables taking values between  $-\pi$  and  $\pi$ .

The algebra in eq.(28) suggests the expressions

$$\begin{aligned} \alpha_3 &= \frac{g\sqrt{L}}{2}\mathcal{L}_3, \\ \alpha_8 &= -\frac{g\sqrt{L}}{2\sqrt{3}}[\mathcal{L}_3 - 2\mathcal{L}_8], \end{aligned} \quad (44)$$

$\mathcal{L}_3$  and  $\mathcal{L}_8$  being the angular momenta conjugate to  $\theta_3$  and  $\theta_8$  respectively.

Eqs.(33) and (44) then provides the spectrum of the hamiltonian

$$E_{n_3, n_8} = \frac{g^2 L}{6}(n_3^2 - n_3 n_8 + n_8^2), \quad n_3, n_8 \in \mathbb{Z}. \quad (45)$$

We remark that, apart from the fundamental level ( $n_3 = n_8 = 0$ ), all other energy eigenvalues are linearly increasing with  $L$  and would diverge in the decompactification limit.

A unique vacuum corresponds to  $n_3 = n_8 = 0$  [2].

### III. WILSON LOOPS

Recently perturbative Wilson loop calculations in 1+1 dimensions gave different results according to different expressions for the propagator [5]. As those results have been obtained

in the continuum, we feel important to reexamine them on the cylinder, also in view of the recent abundant literature on  $QCD_2$ .

In order to avoid an immediate interplay with topological features, we consider a Wilson loop entirely contained in the basic interval  $-L \leq x \leq L$ . We choose a rectangular Wilson loop  $\gamma$  with light-like sides, directed along the vectors  $n_\mu$  and  $n_\mu^*$ , with lengths  $\lambda$  and  $\tau$  respectively, and parametrized according to the equations:

$$\begin{aligned} C_1 : x^\mu(s) &= n^\mu \lambda s, \\ C_2 : x^\mu(s) &= n^\mu \lambda + n^{*\mu} \tau s, \\ C_3 : x^\mu(s) &= n^{*\mu} \tau + n^\mu \lambda (1 - s), \\ C_4 : x^\mu(s) &= n^{*\mu} \tau (1 - s), \quad 0 \leq s \leq 1, \end{aligned} \tag{46}$$

with  $\lambda + \tau < 2\sqrt{2}L$ .

We are interested in the quantity

$$W(\gamma) = \langle \mathbf{0} | \text{Tr} \mathcal{TP} \left( \exp \left[ ig \oint_\gamma A dx^+ \right] \right) | \mathbf{0} \rangle, \tag{47}$$

where  $\mathcal{T}$  means time-ordering and  $\mathcal{P}$  color path-ordering along  $\gamma$ .

The vacuum state belongs to the physical Hilbert space as far as the non vanishing frequency parts are concerned; it is indeed the Fock vacuum  $|\Omega\rangle$ . Then we consider its direct product with the lowest eigenstate of the Hamiltonian in eq.(33) concerning zero modes. This eigenstate belongs to the domain of  $a_0$ ; however, when  $b_0$  acts on it, it generates new states which are no longer in the domain of  $a_0$ , as is well known from elementary quantum mechanics. As a consequence, due to zero modes, we cannot define a “bona fide” propagator for our theory: this should not come to a surprise in view of the topological nature of  $b_0$ . On the other hand a propagator is not required in eq.(47).

We shall first discuss the simpler case of QED, where no color ordering is involved. Eq.(47) then becomes

$$W(\gamma) = \langle \mathbf{0} | \mathcal{T} \left( \exp \left[ ig \oint_\gamma A dx^+ \right] \right) | \mathbf{0} \rangle, \tag{48}$$

and a little thought is enough to realize the factorization property

$$W(\gamma) = \langle \mathbf{0} | \mathcal{T} \left( \exp \left[ \frac{ig}{\sqrt{2L}} \oint_{\gamma} (b_0 + a_0 t) dx^+ \right] \right) | \mathbf{0} \rangle \langle \mathbf{0} | \mathcal{T} \left( \exp \left[ ig \oint_{\gamma} \hat{A}(t, x) dx^+ \right] \right) | \mathbf{0} \rangle = W_0 \cdot \hat{W}, \quad (49)$$

according to the splitting in eq.(11). In turn the Wilson loop  $\hat{W}$  can also be expressed as a Feynman integral starting from the lagrangian in eq.(1) for QED, without the zero mode

$$\hat{W}(\gamma) = \mathcal{N}^{-1} \left( \exp \left[ g \oint_{\gamma} \frac{\partial}{\partial J} dx^+ \right] \right) \left[ \int \mathcal{D}\hat{A} \mathcal{D}\lambda \exp i \left( \int d^2x (\mathcal{L} + J\hat{A}) \right) \right] \Big|_{J=0}, \quad (50)$$

$\mathcal{N}$  being a suitable normalization factor.

Standard functional integration gives

$$\begin{aligned} \hat{W}(\gamma) = \mathcal{N}^{-1} & \left( \exp \left[ g \oint_{\gamma} \frac{\partial}{\partial J} dx^+ \right] \right) \\ & \exp \left[ i \iint d^2\xi d^2\eta J(\xi) \hat{G}_c(\xi - \eta) J(\eta) \right] \Big|_{J=0}. \end{aligned} \quad (51)$$

Using the equations of motion, one can verify that  $\hat{G}_c$  obeys the free inhomogeneous hyperbolic differential equation

$$\partial_-^2 \hat{G}_c(t, x) = \delta(t) \left[ \delta(x) - \frac{1}{2L} \right], \quad (52)$$

with causal boundary conditions.

The canonical algebra provides us with the solution

$$\hat{G}_c(t, x) = \frac{|t|}{2L} \left[ \theta(t) \sum_{n=1}^{\infty} \exp \left( \frac{-i\pi n}{L} (t + x) \right) + \theta(-t) \sum_{n=1}^{\infty} \exp \left( \frac{i\pi n}{L} (t + x) \right) \right]. \quad (53)$$

The causal nature of positive and negative frequency treatment is clearly exhibited.

Eq.(53) should be compared to the expression for the “potential” used by ’t Hooft [6]

$$\hat{G}(t, x) = \frac{|t|}{4L} \sum_{n=-\infty, n \neq 0}^{\infty} \exp \left( \frac{i\pi n}{L} (t + x) \right) = \frac{1}{2} |t| \left[ \delta_P(t + x) - \frac{1}{2L} \right], \quad (54)$$

which coincides with its real part. Eq.(54) is indeed the Green’s function one obtains when interpreting  $\partial_-^2$  not as a propagation, but rather as a constraint. One can also easily check the relation

$$\hat{G}_c(t, x) = \hat{G}(t, x) - \frac{it}{4L} P \text{ctg}\left[\frac{\pi}{2L}(t+x)\right], \quad (55)$$

where  $P$  means Cauchy principal value. Eq.(55) is the explicit causal expression of the integral operator  $\hat{G}_c$ . It looks like a complex “potential” kernel, the absorptive part being related to the presence of ghost-like excitations, which are essential to recover the M-L prescription in the decompactification limit  $L \rightarrow \infty$

$$\hat{G}_c \rightarrow -\frac{it}{2\pi} \frac{1}{(t+x) - i\epsilon \text{sign}(t)} = -\frac{it}{2\pi} \frac{1}{(t+x) - i\epsilon \text{sign}(t-x)}. \quad (56)$$

We remark that zero modes are irrelevant in the limit  $L \rightarrow \infty$ .

Now, introducing eq.(55) in eq.(51), we get the expression

$$\begin{aligned} \hat{W}(\gamma) &= \exp\left[i g^2 \oint_{\gamma} dx^+ \oint_{\gamma} dy^+ \hat{G}_c(x^+ - y^+, x^- - y^-)\right] \\ &= \exp\left[i g^2 \oint_{\gamma} dx^+ \oint_{\gamma} dy^+ \hat{G}(x^+ - y^+, x^- - y^-)\right] \\ &= \exp\left[-i \frac{g^2 \mathcal{A}}{2}\right] \exp\left[-i g^2 \oint_{\gamma} dx^+ \oint_{\gamma} dy^+ \frac{|x^+ + x^- - y^+ - y^-|}{4L\sqrt{2}}\right] \end{aligned} \quad (57)$$

the absorptive part of the “potential” averaging to zero. The quantity  $\mathcal{A} = \lambda\tau$  is the area of the loop. The same result can also be obtained by operatorial techniques, using Wick’s theorem and the canonical algebra.

We are thereby left with the problem of computing  $W_0$

$$W_0(\gamma) = \langle \mathbf{0} | \mathcal{T} \left( \exp\left[\frac{ig}{\sqrt{2}L} \oint_{\gamma} (b_0 + a_0 t) dx^+\right] \right) | \mathbf{0} \rangle. \quad (58)$$

The expansion of the exponential in (58) is delicate and, if not carefully treated, could give rise to ambiguous results. We recall that, when  $b_0$  acts on the vacuum state, it generates a state no longer belonging to the domain of  $a_0$ . This problem can be overcome in the following way. Using the canonical algebra, the  $\mathcal{T}$ -product of  $n$  factors in the expansion of (58), can be recursively ordered, by generalizing Wick’s theorem, in a sum of operators in which  $a_0$  factors appear on the right or do not appear at all. Terms without  $a_0$  are in turn of two kinds: either they exhibit some  $b_0$ ’s or are  $c$ -number products of contractions. Only those last terms provide non vanishing contributions. As a matter of fact terms with unpaired

$b_0$ 's vanish upon the related  $\oint_\gamma dx^+$ -integration already at the operator level, whereas terms containing  $a_0$ 's vanish when acting on the vacuum on the right [11].

Contraction terms are unambiguous and obviously entail an even number of fields; thus the  $2n$ -th term of the expansion provide us with  $(2n - 1)!!$  expressions of the kind

$$\frac{ig^2}{2L} \oint_\gamma dx_j^+ \oint_\gamma dx_k^+ \max\{t_j, t_k\} = i g^2 \oint_\gamma dx^+ \oint_\gamma dy^+ \frac{|x^+ + x^- - y^+ - y^-|}{4L\sqrt{2}}. \quad (59)$$

Summing over  $n$  we recover exponentiation with a factor which *exactly cancels* the last exponential in eq.(57), leaving the pure loop area result, only as a consequence of the canonical algebra, and even in the presence of a topological degree of freedom. The result exactly coincides with the one we would have obtained introducing in eq. (57) the complete Green's function, *i.e.* with the zero mode included.

However eq.(57) unfortunately is unable to discriminate between the two different Green's functions  $\hat{G}_c$  and  $\hat{G}$ .

The generalization to the non abelian case is far from being trivial, owing to an intertwining between space and group algebra variables. In particular topological excitations get inextricably mixed with the frequency parts. The very idea of combining zero mode contribution and frequency contractions into an “effective” propagator is unjustified in our opinion. Yet it is true that in the limit  $L \rightarrow \infty$  zero modes do not contribute. Should we close our eyes and try to evaluate the Wilson loop by summing perturbative exchanges in the form  $G(t, x) = \frac{1}{2}|t|\delta_P(t + x)$ , as suggested by the QED case, eq.(47) could be explicitly evaluated, thanks to the “delta-like” character of the “potential”. We would indeed obtain

$$W(\gamma) = Tr \mathcal{P} \left( \exp \left[ g \int_{C_4} \frac{\partial}{\partial J^+} dx^+ \right] \right) \mathcal{P} \left( \exp \left[ g \int_{C_2} \frac{\partial}{\partial J^+} dx^+ \right] \right) \exp \left[ i Tr \iint d^2\xi d^2\eta J^+(\xi) G(\xi - \eta) J^+(\eta) \right] \Big|_{J^+=0}, \quad (60)$$

namely

$$W(\gamma) = Tr \mathcal{P} \left( \exp \left[ g \int_{C_4} \frac{\partial}{\partial J} dx \right] \right) \mathcal{P} \left( \exp \left[ \frac{ig\lambda}{2} \int_{C_4} J dx \right] \right) \Big|_{J=0}, \quad (61)$$

and finally

$$W(\gamma) = \exp\left[-i \frac{g^2 N C_F \lambda \tau}{2}\right], \quad (62)$$

$C_F$  being the quadratic Casimir of the fundamental representation of  $su(N)$ . The area ( $\mathcal{A} = \lambda \tau$ ) law behaviour of the Wilson loop we have found in this case together with the occurrence of a simple exponentiation in terms of the Casimir of the fundamental representation, is a quite peculiar result, insensitive to the decompactification limit  $L \rightarrow \infty$ . It is rooted in the particularly simple expression for the “potential” we have used that coincides with the one often considered in analogous Euclidean calculations [12].

However canonical quantization suggests that we should rather use the propagator  $G_c(t, x) = \frac{1}{2}|t|\delta_P(t+x) - \frac{it}{4L}P \operatorname{ctg}\left[\frac{\pi}{2L}(t+x)\right]$  (see eq.(55)). Then a full resummation of perturbative exchanges would be no longer possible, owing to the presence of non vanishing cross diagrams. Already at  $\mathcal{O}(g^4)$ , a tedious but straightforward calculation of the sum of all the “cross” diagrams in fig. 1 leads to the result

$$W_{cr} = \left(\frac{g^2}{4\pi}\right)^2 2 C_F C_A(\mathcal{A})^2 \int_0^1 d\xi \int_0^1 d\eta \log \frac{|\sin \rho(\xi - \eta)|}{|\sin \rho \xi|} \log \frac{|\sin \rho(\xi - \eta)|}{|\sin \rho \eta|}, \quad (63)$$

where  $\rho = \frac{\pi \lambda}{\sqrt{2}L}$ . One immediately recognizes the appearance of the quadratic Casimir of the adjoint representation ( $C_A$ ); moreover, besides the area dependence, a dimensionless parameter  $\rho$ , which measures the ratio of the loop length  $\lambda$  to the interval length  $L$ , explicitly occurs. In the decompactification limit  $\rho \rightarrow 0$ , the expression [5]

$$W_{cr} = \left(\frac{g^2}{4\pi}\right)^2 C_F C_A(\mathcal{A})^2 \frac{\pi^2}{3} \quad (64)$$

is smoothly recovered. While the presence of zero modes puts severe doubts on any perturbative calculation owing to the poor definition of propagators, it is perhaps not surprising that in the limit  $L \rightarrow \infty$  the perturbative result in the continuum is correctly reproduced.

#### IV. CONCLUDING REMARKS

To conclude, we briefly summarize our results. We have succeeded in canonically quantizing at equal **time** in light-cone gauge pure Yang-Mills theories, defined on the space-time

$S_1 \times R$  cylinder. Gauss' law is imposed in a weak sense, e.g. as a condition on the Fock space vectors. Creation and annihilation operators, corresponding to non vanishing frequencies, do not contribute in the “physical” Hilbert space; nevertheless they give rise to a propagator, which is the light-cone counterpart of the Coulomb interaction term in the Coulomb gauge. Equal time quantization induces a causal behaviour of this propagator, making it different from the “contact” expression often considered in the literature. In the decompactification limit  $L \rightarrow \infty$ , it naturally becomes the causal M-L distribution.

Zero modes are present as topological degrees of freedom and quantized according to their winding around the cylinder.

A Wilson loop with light-like sides is exactly computed in QED; the area law is recovered even in the presence of topological degrees of freedom.

In the non Abelian case a perturbative approach would be hindered by the poor definition of a propagator in the presence of topological excitations. In addition the causal nature of the sum over frequency contributions prevents an exact solution; a perturbative solution at  $\mathcal{O}(g^4)$  in the limit  $L \rightarrow \infty$  reproduces smoothly the result we have previously found in the continuum [5].

Another way of understanding the difficulties of the Wilson loop calculation is to observe its equivalence with a fermionic problem: it is well-known [13] that the two-point function for dynamical one-dimensional fermions living on a loop and interacting with a Yang-Mills theory, defined on a manifold containing the loop itself, *exactly* gives (at coincident points) the value of the related Wilson loop for pure Y-M theory. In this sense, due to the presence of interactions between these auxiliary fermions, frequencies come into game, mixing with the (topological) gauge invariant degrees of freedom. We remark that the treatment of loops winding around the cylinder is somewhat easier to handle than of the non-wrapped ones because they are directly related to the quantum mechanical degrees of freedom: large-N computations for such a class of observables are presented in [14]. On the other hands it was shown in [15] that the presence of Wilson lines (described by the formalism of one-dimensional fermions) turns the theory into an effective Calogero-Sutherland model [16]:

it could be very insightful to understand if in our case the exact evaluation of zero-modes contributions to the Wilson loop corresponds to the solution of some suitable quantum mechanical model.

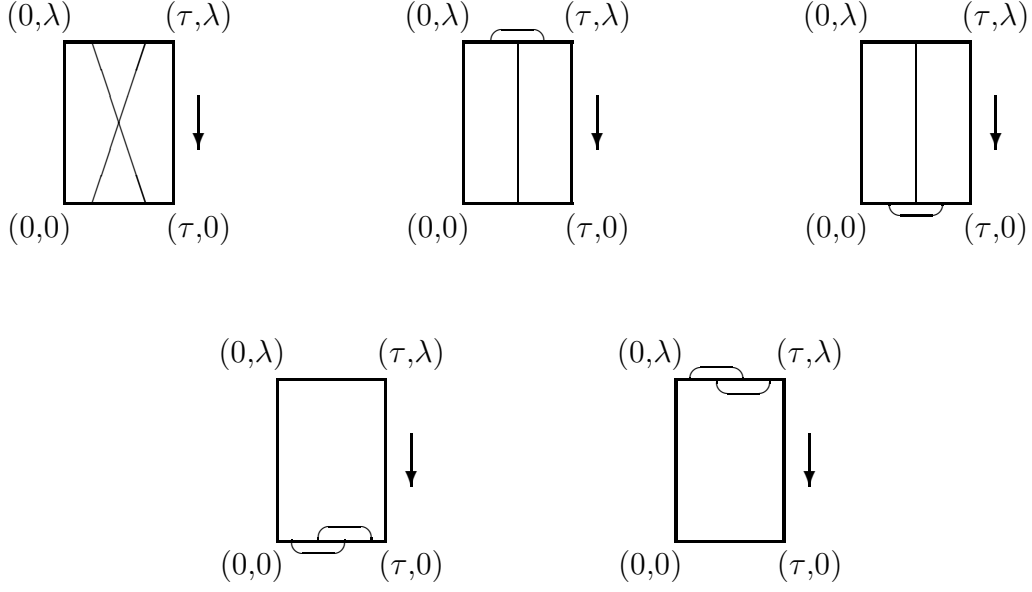
Other interesting issues are open to future investigation, the most exciting one perhaps concerning the introduction of dynamical (two-dimensional) massless fermions.

Finally it should be worth understanding to what extent the features we have found on the 1+1 cylinder, may apply to the more difficult , but realistic case of 3+1 dimensions.

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**Fig. 1**

#### FIGURE CAPTIONS

**Fig. 1:** Crossed diagrams contributing with a  $C_F C_A$  term to the  $g^4$  perturbative evaluation of the Wilson loop with causal propagator  $G_c(t, x)$ . Thick lines denote the rectangular loop in the  $x^+ \times x^-$  plane, with sides length  $\tau$  and  $\lambda$ , respectively. Thin lines denote gluon propagators.